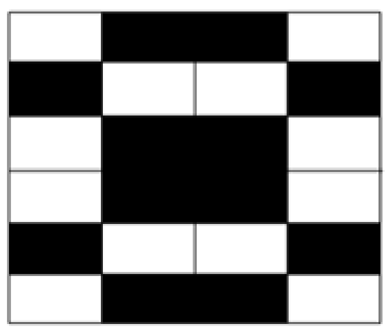
2.2 Pattern Recognition: Searching for Connections

"The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics." —Godfrey Hardy

Introduction

Now that you've begun to put your logical and creative thinking skills into action, this section directs you to use your powers of observation to examine and explore numerical patterns of various sorts. In the process of doing so, ***you will come to appreciate the usefulness of some basic mathematical tools including pattern recognition, asking questions and making conjectures.***



Pattern Recognition

Let's begin with some very simple patterns that we all can easily recognize. For example,

2, 4, 6, 8, …

is a most familiar pattern, one that you probably learned in kindergarten or in early childhood. We all know that the next number in this sequence of numbers is 10. A ***mathematical sequence***, by the way, ***is simply an ordered list of elements; these elements are often referred to as "terms."***

******** Opportunities to Exercise your Analytical Reasoning Skills ********
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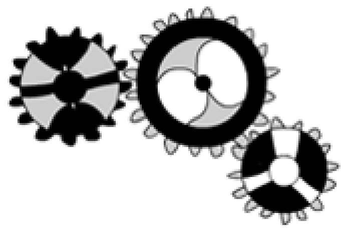
Some other examples are:

3, 7, 11, 15, 19, … and 14, 23, 32, 41, 50, …

With a bit of observation, I'm sure that you can determine the next terms for each of the above sequences. Please take a few moments to do this and put your mental gears in motion. (Answers to these and succeeding examples are provided on p.64 at the end of this section.)



Here is another: 2, 6, 18, 54, …

This, too, while different in nature from the first two examples, yields its secret to a bit of observation and analysis. Please use your powers of observation and creative thinking to determine the next term; this activity is designed to actively engage you in your reading and help to build your confidence in your analytical reasoning.

And another: 12, 6, 3, 1.5, …

Please give this one a try, too. Although the next term may not be immediately obvious, a bit of effort and persistence on your part will lead to a very satisfying conclusion. Since mathematical skills are developed incrementally, these opportunities to exercise your skills are valuable elements in your mathematical growth so please don't ignore them!

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Hopefully, you are meeting with a measure of success already!

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All of these examples involve the processes of analyzing relationships between the successive numbers in the sequences and determining a common feature among all of the relationships. These are excellent mathematical skills to develop as they may be applied to many other areas in life.

As you may have noticed, the first three examples involved ***addition*** and exhibited a ***common difference*** between terms while the latter two examples involved ***multiplication*** and exhibited a

common ratio (or common multiplier) between terms. Technically, the first three are known as ***arithmetic sequences*** and the latter two are known as ***geometric sequences.*** These two types of sequences are quite prevalent in life; we will encounter them again later in the textbook.

> ***Note:*** When used as an adjective, *arithmetic* is pronounced *ă'-rĭth-***mĕ'***-tĭc* with the heavier accent on the syllable *me*.

While these two types of sequences are rather common, they are by no means the only types; there is an infinite variety of sequences, limited only by one's imagination.

Here are a few examples; the next terms are rather obvious.

7, 7, 7, 7, 7, …	constant sequence
1, –1, 1, –1, 1, …	alternating sequence
1, 1, 2, 2, 3, 3, …	sequence of doublets
1, a, 2, b, 3, c, …	alphanumeric sequence

Successive Differences: An Analytical Tool

As you may have already discovered, examining the differences in value between successive terms in a sequence can be quite informative. In the ***arithmetic sequences*** above, the difference between successive terms is the key to constructing the next terms in the sequence. Since this difference is the same for each pair of terms, we call this difference a ***common*** difference and simply ***ADD this common difference*** to a given term in order to construct the next term.

In the sequence,

5, 17, 29, 41, …

we can easily determine the differences between successive terms by subtracting a given term from the next term in the sequence:

17–5=12, 29–17=12, 41–29=12, and so on.

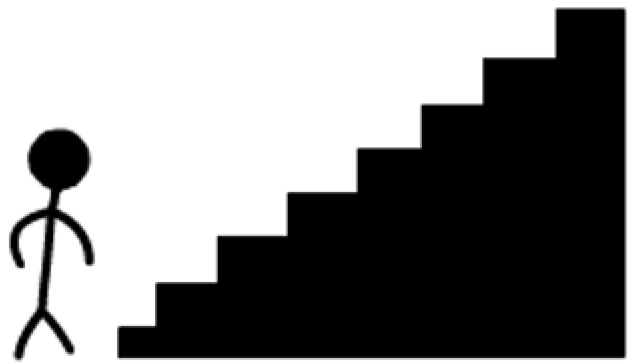
Since each difference is 12, we say that this sequence has a ***common*** difference (i.e., the difference is common to each pair of successive terms). The next term is found by simply adding this common difference to the previous term: 41+12=53.

sequence terms:	5,	17,	29,	41,	53, …
common differences:	12	12	12	12	

As another example, in the sequence,

9, 16, 23, 30, …

there is a common difference of 7 and the next term is simply 30+7=37. *Arithmetic* sequences, due to their very simple structure, are easy to work with and we'll examine them in greater depth in the next section on mathematical generalization.



A Different Type of Sequence

In the following sequence, things are a little less simple:

1, 4, 9, 16, 25, …

***** ***Opportunities to Exercise your Analytical Reasoning Skills*** *****
☺☺☺

Can you determine what the next term should be? Did you discover a common difference between the terms? Please take a moment to exercise your reasoning skills and answer these two questions; again, your efforts here will build your confidence in your mathematical thinking processes. (Answers may be found on p.64 at the end of this section.)

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This is a case where there was not a common difference between the terms, although the successive differences themselves exhibited a pattern. It is often instructive to apply our analysis tool again to see what it might reveal. In the case of the successive differences (found by subtracting a given term from the one that follows it in the sequence),

sequence terms:	1, 4, 9, 16, 25, …
successive differences:	3, 5, 7, 9, …

there is an obvious pattern: the differences are increasing at a steady rate. If we examine the successive differences (again found by subtraction) in the secondary sequence, 3, 5, 7, 9, …

we find that these terms are related by a ***common difference of 2:***

5–3=2, 7–5=2, 9–7=2, and so on…

Ultimately, then, the original sequence depends on the value 2 for its structure; 2 is somehow inherent to the nature of this sequence of positive integer squares, i.e., 1^2, 2^2, 3^2...
Interesting, isn't it?

sequence terms:	1, 4, 9, 16, 25…
successive differences:	3, 5, 7, 9…
common differences:	2 2 2

Digging Deeper

Let's take a look at another sequence:

1, 8, 27, 64, 125, …

******** Opportunities to Exercise your Analytical Reasoning Skills ********
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Can you determine what the next term should be? Did you discover a common difference between the terms? (Answers may be found on p.64 at the end of this section.)

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This is another case where there is not a common difference between the terms of the sequence although the successive differences themselves may contain a pattern. So we'll apply our analysis tool again and look at the differences between the successive differences.

sequence terms:	1, 8, 27, 64, 125, …
successive differences:	7, 19, 37, 61, …
second successive differences:	12, 18, 24, …

This deeper sequence,

12, 18, 24, …

reveals a definite pattern and we can apply our analysis tool of successive differences once more. In other words, we'll examine the differences of the differences of the differences. Whew! That was a mouthful, but it was fun to say; please forgive my fit of madness. ☺

second successive differences: 12, 18, 24…
common differences: 6 6

In this case, we arrive at a constant common difference of 6, much as we arrived at a constant common difference of 2 in the previous example involving whole number squares. Just as with the squares, we can see that there is a number, 6, that is somehow inherent to the nature of the sequence of positive integer cubes, i.e., 1^3, 2^3, 3^3... Very interesting, indeed!

While all students are familiar with integer squares and cubes, very few students have investigated the underlying structures of these numbers.

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In the preceding investigation, as brief as it was, we were able to discover some previously hidden relationships and come away with a sense that there are some deeper connections here that may be interesting to explore; in fact, this may be the beginning of a real mathematical adventure!

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You'll have opportunities to do some of this exploring in the *Exercises and Projects for Fun and Profit* at the end of this section and to make some of your own personal discoveries.

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Reverse Gear

Let's return to the sequence of cubes and the layers of successive differences as noted above. By beginning with the constant sequence of 6's and working in reverse, we can construct the next terms in each of the layers of successive differences and eventually arrive at the original sequence and construct its next term. Pretty clever, eh?

To construct the next term in the layer above the sequence of constant 6's, we add 24 plus the common difference of 6 to get 30:

1, 8, 27, 64, 125
7, 19, 37, 61
12, 18, **24**, **30** **30 = 24 + 6**
6 6 **6**

To go up another level, we add 61 plus the successive difference of 30 to get 91:

1, 8, 27, 64, 125
7, 19, 37, **61, 91** **91 = 61 + 30**
12, 18, 24, **30**
6 6 6

To reach the top level of the original sequence we add 125 plus the successive difference of 91 to get 216:

1, 8, 27, 64, **125, 216** **216 = 125 + 91**
7, 19, 37, 61, **91**
12, 18, 24, 30
6 6 6

This reversing process may be repeated as often as desired to construct more and more terms of the original sequence (although in this case it is much easier to directly calculate the integer cubes). In conclusion, the underlying structure of the original sequence has been analyzed to reveal its basis and, in turn, this basis has been used to extend the original sequence.

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This is a very effective and powerful form of reasoning and analysis that may be applied in various and sundry situations both in mathematics and in many other fields.

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Conjectures: Creative Guesswork

Now that we have discovered underlying common differences for the sequences of squares and of cubes, we have before us a wonderful opportunity to engage in a very rich and fruitful area of mathematics: ***formulating questions and making conjectures based on a given set of facts or observations.*** These questions can be of infinite variety, but often take the form of:

- It appears as though there is a pattern. I wonder whether the pattern continues?
- What if the facts or situation were slightly different? How would that affect the outcomes?
- Why is that so?
- Why not?

Mathematicians (and students when they are encouraged ☺) often engage in this form of activity and it has been the basis for many exciting discoveries and rewarding experiences. I've taken the liberty to recoin an old saying: ***"You learn something new every day, and every so often you make a discovery."***

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> ***While learning is delightful, the personal discoveries that you make add richness, depth and joy to your experience of mathematics and bring a sense of newness to your relationship with math.***

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One of my aims in writing this textbook was to lead you to make personal discoveries of this sort and you will have numerous opportunities to do so, including the one before you now.

******** An Opportunity to Exercise your Questioning and Conjecturing Skills ********
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With regard to the underlying common differences for the sequences of squares and of cubes that we discovered, 2 for the squares and 6 for the cubes, you may now perform a simple yet very valuable mathematical activity: ask some related questions of interest and make your own mathematical conjectures. Since a ***conjecture*** is simply creative guesswork, everybody can join in and have fun!

Please take some time now and make a list of creative questions and/or conjectures (minimum of two) concerning our observations concerning underlying differences for squares and for cubes; these questions and/or conjectures are of great value in terms of increasing your mathematical creativity. ***This is an open-ended activity and one in which there are no "wrong" answers, so relax and enjoy the process!***

> ***Note:*** Some questions and conjectures are provided on p.64 at the end of this section after the *Exercises and Projects for Fun and Profit,* but please refrain from reading them until you have made your own—the process of making them is just as important as the results themselves!

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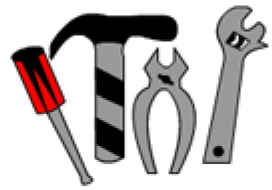
Rather than investigate the related questions and conjectures at the moment, we'll include these as *Further Investigations* at the end of the section ***where you may practice your newly acquired analysis tool, successive differences.***

Insights and Conclusion

In this section, you began to develop your skills in formulating questions and making conjectures based on a given set of facts or observations; these skills are quite valuable and are applicable in many fields.



We began with some simple number sequences to exercise your skills in ***pattern recognition,*** a basic and fundamental mathematical endeavor. After encountering several types of sequences, we presented an analytical tool called ***successive differences*** to enable you to see underlying patterns more easily.

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> ***Simple mathematical tools such as this often give us access to otherwise vague and murky areas; we'll develop a number of these tools throughout the remainder of this textbook.***

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Using this analysis tool, we were able to probe the depths of the nature of positive integer squares and cubes and discover some quite interesting and unexpected features. This is the type of investigation that can lead to new and unusual discoveries, even breakthroughs of sorts. ***It's quite exciting to explore a mathematical topic at this depth—it's almost like being on a new frontier!***

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Exercises and Projects for Fun and Profit

Exercises in Analysis

> ***Note:*** Please see ***Helps and Hints for Exercises*** on **pp.442–433 *in Appendix A*.**

What Comes Next?
Use your powers of observation to determine the next terms in each of the following sequences:

1. 4, 19, 34, ___, …

2. 17, 28, 39, ___, …

3. 42, 70, 98, 126, ___, …

4. 33, 52, 71, 90, ___, …

5. 2, 6, 18, ___, …

6. 16, 36, 81, ___, …

7. 81, 27, 9, ___, …

8. 125, 50, 20, ___, …

Missing Terms and Analytical Reasoning

Use your analytical reasoning skills to find the missing term in each of the following sequences:

9. 12, 19, ___, 33, …

10. 2, ___, 20, 29, …

11. ___, 2, 8, 32, 128, …

12. 6, 15, ___, 93.75, …

Interwoven Sequences: Two for the Price of One

Exhibiting a technique used on occasion by cryptographers, the following sequences consist of two sequences woven together. The effect of the interweaving is to make pattern recognition more difficult in that the weaving method is not always obvious. In the case of alphanumeric sequences such as #13–14 below, the weaving technique is clear; however, in some of the other exercises you'll have to search more diligently to find the correct weave. Use your analytical reasoning skills to find the ***next two terms*** in each of the following sequences:

13. d, 17, g, 23, j, 29, ___, ___, …

14. a, 12, f, 21, k, 30, p, ___, ___, …

15. 2, 9, 4, 13, 6, 17, ___, ___, …

16. 4, 2, 8, 4, 12, 6, ___, ___, …

Interwoven Sequences: Three at Any Price

Use your analytical reasoning skills to find the ***next three terms*** in each of the following sequences in which three sequences have been woven together. You will need to make an initial conjecture concerning how the three sequences are woven together; the simplest conjecture will be the most effective…

17. 5, 8, 10, 11, 14, 7, 17, 20, 4, ___, ___, ___, …

18. 20, 17, 9, 13, 14, 11, 6, 11, 13, ___, ___, ___, …

19. 7, a, z, 15, b, y, 23, c, x, ___, ___, ___, …

20. 2, 6, f, 8, 15, k, 14, 24, p, ___, ___, ___, …

A Mixed Bag: The Challenge of the Unexpected

In this miscellaneous collection of sequences, use your powers of observation and critical thinking skills to find the ***missing terms.*** In each of these cases, you'll need to make initial conjectures as to how many sequences are involved and in what way they are woven together.

21. 5, 3, 9, 7, 13, 11, ___, ___, …

22. 17, 13, 20, 16, 23, 19, 26, ___, ___, …

23. 9, –6, 14, –1, 19, ___, ___, …

24. 5, 8, 13, 11, 14, 19, 17, ___, ___, ___, …

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Further Investigations

25. ***A fourth dimension:*** Following the examples given in the text for sequences of squares and cubes, use your computation skills (viz., subtraction) and investigative reasoning to determine the layers of successive differences and, ultimately, the underlying ***common difference*** for the sequence of numbers representing positive integer fourth powers. The sequence of positive integers raised to their fourth powers is:

1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000…

> ***Warning:*** You'll need to be very careful with your arithmetic here; any subtraction errors may cause the whole process to turn out to be hopelessly indecipherable. ☹ ***It would be most advantageous to use a calculator….***
>
> ***Hint:*** You'll need to calculate several layers of successive differences before you finally discover the row in which the differences are constant.

26. ***A fifth dimension:*** Following the examples given in the text for sequences of squares and cubes, use your computation skills (viz., subtraction) and investigative reasoning to determine the layers of successive differences and, ultimately, the underlying ***common difference*** for the sequence of numbers representing positive integer fifth powers. The sequence of positive integers raised to their fifth powers is:

1, 32, 243, 1024, 3125, 7776, 16807, 32768, 59049, 100000…

> ***Warning:*** You'll need to be very careful with your arithmetic here; any subtraction errors may cause the whole process to turn out to be hopelessly indecipherable. ☹ ***It would be most advantageous to use a calculator….***
>
> ***Hint:*** You'll need to calculate several layers of successive differences before you finally discover the row in which the differences are constant.

27. ***Dimensional patterns:*** Record the underlying common differences from the preceding investigations with squares, cubes, and fourth and fifth powers. Construct a table of these values as shown below in order to more clearly see possible relationships—***a table (or chart) is often a great help in recognizing patterns because it organizes the data in a very simple and basic structure and eliminates unnecessary verbiage:***

type of number	***power***	***underlying common difference***
square	2	2
cube	3	6
fourth power	4	
fifth power	5	
sixth power	6	

Do you observe a pattern in the common differences of these various powers (second power=squares, third power=cubes, fourth power, fifth power)? If so, what do you see?

Based on your observation, make ***a conjecture concerning the underlying common difference for sixth power numbers.***

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> ***Searching for patterns is a fundamental mathematical activity, one that finds application in many areas and sometimes leads to new and exciting discoveries.*** While the above investigation may not seem exciting to you (after all, who cares much about the sixth powers of positive integers), if you have successfully discovered the pattern involved and made a correct conjecture about the underlying common difference for sixth powers, then ***you may give yourself a hearty congratulations and a big pat on the back! You are already beginning to expand and improve your analytical reasoning skills!***

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Creative Projects:

The Curious Date: 7-14-98
Another Curious Date: 10-1-01

THE CURIOUS DATE: 7-14-98

I was attending a workshop one summer and for some reason needed to record the date, July 14, 1998. As I wrote the standard date notation, 7-14-98, on my paper, I was struck by the curious nature of the numerical values contained in the date. Do you see what I mean? Since I had never noticed this before, I wondered whether this was because this sort of occurrence was relatively rare. Or perhaps I had not been very observant in the past….

<u>Assignments</u>:

This observation immediately led to a number of related questions that now comprise this project**.** If you need help in determining the curious connection above, please see *Note* below. Afterwards, for each assignment ***please consider each of the ten years in the indicated decade (ten-year period) and answer the seven questions below:***

assignment	***years***
Assignment #1	1990–1999, inclusive
Assignment #2	1980–1989, inclusive
Assignment #3	1970–1979, inclusive
Assignment #4	1960–1969, inclusive

1. Which year in the given decade had the most such unusual occurrences?
2. Which year(s) in the given decade had the next largest number of such occurrences?
3. Which years in the given decade had exactly one such occurrence?
4. Which years in the given decade had no such occurrences?
5. How many curious dates were there in the given decade?
6. After today (the day on which you are doing this project), when will the next such occurrence be?
7. So, then, just how rare was this sort of occurrence (i.e., what percentage of dates in the given decade had this special property)? To find out, divide the number of curious dates by the total number of days in a decade (a ten-year period).

****

To help you recognize curious dates, here are some examples from <u>other</u> decades:

1900: no such occurrences
1920: 1-20-20, 2-10-20, 4-5-20, 5-4-20, and 10-2-20 are curious dates
1939: 3-13-39 is the only curious date
1952: 2-26-52 and 4-13-52 are curious dates

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Note: In case you haven't made the connection yet, the curious fact is that the month multiplied by the day equals the last two digits of the year: 7 · 14 = 98. ☺

ANOTHER CURIOUS DATE: 10-1-01

You'll never guess what happened one morning at breakfast: my wife noticed that the date was very curious! We had been commenting that the day was the first day of October when she said, ***"Do you know that today's date reads the same backwards as it does forwards?"***

And sure enough it did! ***10-1-01***

This got our whole family thinking about whether this was a rare occurrence or not and we realized that there were at least *some* other dates with this property but didn't have time to pursue the matter further before our schedules required us to move on.

Please investigate this new curious date phenomenon for yourself. The only requirement is that the numbers in the date read the same forwards and backwards. In particular, the hyphen position doesn't matter, and, as per our usual convention in writing dates by hand, dates like 01-11-10 or 3-09-03 with leading zeros in the month and/or day positions will not be considered (although this could be investigated in a separate project).

Also, we will restrict ourselves to the two-digit format for the year rather than the four-digit format; consequently, you don't need to consider dates in the form 10-9-1901

> ***As an example,*** the date 3-2-23 would be considered curious because the numbers 3223 (ignoring the hyphens) read the same forwards (3223) and backwards (3223). For the record, this is called a ***palindromic number***. (A ***palindrome*** is a word or expression that reads the same both forwards and backwards, e.g., toot, madam, solos, etc.)
>
> Other examples are: 12-8-21 (12821), 9-29-29 (92929), 5-7-75 (5775), etc.
>
> ***A non-example is:*** 2-3-23. This date is NOT curious because it doesn't read the same both forwards (2323) and backwards (3232).

<u>Assignments</u>:
For the years indicated in each assignment, please answer the three questions below:

assignment	*years*
Assignment #1	1920–1929, inclusive
Assignment #2	1930–1939, inclusive
Assignment #3	1940–1949, inclusive
Assignment #4	1950–1959, inclusive

1. Determine how many curious dates of this nature occurred and record them in a list.
2. Report the total number of dates that you found.
3. Determine the percentage of dates in the given period that have this property (i.e., divide the number of curious dates by the number of days in the given ten-year period).

Answers to Questions in this Section

p.50 Pattern Recognition

The next terms are included below:

3, 7, 11, 15, 19, **23**	add 4 to each term
14, 23, 32, 41, 50, **59**	add 9 to each term
2, 6, 18, 54, **162**	multiply each term by 3
12, 6, 3, 1.5, **0.75**	multiply each term by 0.5, i.e., multiply by $\frac{1}{2}$

p.52 A Different Type of Sequence

1, 4, 9, 16, 25, **36** — the differences are changing at a steady rate; in case you haven't noticed, these numbers are positive integer squares: $1^2, 2^2, 3^2, 4^2, \ldots$

p.53 Digging Deeper

1, 8, 27, 64, 125, **216** — in case you haven't noticed, these numbers are positive integer cubes: $1^3, 2^3, 3^3, 4^3, \ldots$

p.56 Conjectures: Creative Guesswork

Yes, that's it! The most likely topic is:

in question form

- Does the sequence of numbers representing positive integer fourth powers have an underlying common difference?

in conjecture form

- I would surmise at this point that the sequence of numbers representing positive integer fourth powers has an underlying common difference.

Other conjectures (these may also be presented in question form):

- There will be underlying common differences for the sequences of positive integer fourth powers, fifth powers, sixth powers, et al.
- The underlying common difference for the sequence of positive integer fourth powers will be greater than 6.
- Since the underlying common difference for cubes was 6, which is three times the common difference for squares, the underlying common difference for fourth powers might be 8, i.e., four times the common difference for squares.